TRANSCENDENTAL NUMBERS

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By definition, y is an algebraic function of x if it is a function that satisfies an irreducible algebraic equation of the form

$$P_0(x)y^n + P_1(x)y^{n-1} + . . . + P_{n-1}(x)y + P_n(x) = 0,$$

With n a positive integer and with coefficients $P_0(x)$, $P_1(x)$, . . .; that are polynomials in x. A function that is not algebraic is called transcendental. [11, p. 230]

This definition or an equivalent form may be found in a calculus, number theory, analysis or other mathematics text. The last sentence of the definition was responsible for this study. It naturally leads to the innumerable questions concerning the types, classes, and characteristics of functions that are transcendental.

The original intent of this project was to consider the familiar transcendental numbers and functions such as e, π, log x, sin x and to classify numbers of the form a^t, t^a, a^b and ${t_1}^{t_2}$; where a and b are algebraic and the t_i's transcendental. Subsequent research denied the latter objective and dictated consideration of specific numbers such as e^π, 2^π, e^a and e^e.

Although e, π, sin x and log x are familiar concepts and frequently used tools of mathematics, the proof of their transcendence is relatively recent development. In 1873, Charles Hermite proved e is transcendental. [2, p. 464] Hermite's work is generally accepted as the catalyst for all later studies. However, the first proof concerning transcendental numbers was presented only twenty-nine years earlier. Joseph Liouville proved in 1844, that a number of the type

$$1/n + 1/n^2 + 1/n^6 + 1/n^{24} + 1/n^{120} + \ldots,$$

where n is a real number greater than 1, is transcendental. [2, p. 463]

In 1881 as a direct result of Hermite's proof, Ferdinand Lindermann proved π is transcendental. Since that time, Alexis Gelfond presented the only significant generalization in 1934. He proved all numbers of the type a^b where a is neither 0 nor 1 and b is any irrational algebraic number, are transcendental. [2, p. 463]

Proving e is a transcendental number involves a lemma and the theorem proper, both of which are presented by Niven in Irrational Numbers. Following is a statement of the lemma and the theorem with a descriptive analysis.

> LEMMA: If $h(x) = x^n g(x)/n!$ where g(x) is a polynomial with integral coefficients, then $h^j(x)$, the j-th derivative of h(s) evaluated at x=0, is an integer for j = 0, 1, 2, ... Moreover, with the possible exception of the case j = n, the integer $h^j(0)$ is divisible by n+1; no exception need be made in the case j = n if g(x) has a factor x: i.e., if g(0) = 0.
>
> Proof: We may write
>
> $$h^j(0) = c_j(j!)/n!,$$
>
> where c_j is the coefficient of x^j in the polynomial $x^n g(x)$, so that c_j is an integer by hypothesis. For j<n we have $c_j = 0$. For j>n it is apparent that $h^j(0)$ is divisible by n + 1. In case j = n we have $h^n(0) = c_n$ and, if g(0) = 0, then $c_n = 0$. [7, p. 16]

Lemma Analysis: In the theorem, it will be necessary to apply the lemma when $n = p-1$. Consequently, in the following analysis, $n = p-1$.

The statement $h^j(0) = c_j(j!)/n!$, where c_j is an integer, is justified since j^{th} derivative evaluated at 0 implies all x^m, when $m>j$, will have x^{m-j} as a factor and, therefore, will equal zero. In addition, all x^r, where $r<j$, will equal zero since $d(k)/dx = 0$, where k is a constant. Now we have the coefficient of the x^j term, namely $c_j(j!)$; i.e.,

$$c_j[j(j-1)(j-2)\ .\ .\ .]x^0 = c_j(j!).$$

This implies

$$h^j(0) = c_j(j!)/(p-1)!.$$

Now, if $j>(p-1)$, we have

$$h^j(0) = c_j[(p-1) + h]!/(p-1)!$$

$$= c_j[(p-1)(p)(p+1)(p+2).\ .\ .(p+h)]/(p-1)!$$

$$= c_j[p(p+1)(p+2).\ .\ .(p+h)],$$

which implies $h^j(0)$ is a multiple of p and $h^j(0)$ is an integer.

Next, if $j<(p-1)$, then

$$h^j(0) = 0 = c_j,$$

since all terms will contain a factor of x. Finally, if $j = (p-1)$, then

$$h^{p-1}(0) = (p-1)!c_{p-1}/(p-1)!$$

$$= c_{p-1.}$$

Therefore, if g(x) has a factor of x, thus, $g(0) = 0$, then $h^{p-1}(0)$ will have a factor of x and consequently, $h^{p-1}(0) = 0$.

THEOREM 1: e satisfies no relation of the form

$$a_m e^m + a_{m-1}e^{m-1} + . . . + a_1 e + a_0 = 0,$$

having integral coefficients not all zero. (Stated otherwise, e is a transcendental number.)

Proof: There is no loss of generality in presuming that $a_0 \neq 0$. We define

$$f(x) = x^{p-1}(x-1)^p(x-2)^p . . . (x-m)^p/(p-1)!$$

and

$$F(x) = f(x) + f^1(x) + f^2(x) + f^3(x) + . . . + f^{mp+p-1}(x),$$

Where p is an odd prime to be specified. For 0<x<m we have

#1 $$|f(x)| < m^{p-1}m^p m^p . . . m^p/(p-1)! = m^{mp+p-1}/(p-1)!.$$

We readily verify that

$$d[e^{-x}F'x]/dx = e^{-x}[F'(x) – F(x)] = -e^{-x}f(x),$$

and

$$a_j \int_0^j e^{-x}f(x)dx = aj[-e^{-x}F(x)]_0^j = a_jF(0) – a_je^{-j}F(j).$$

We multiply by e^j, then sum over the values j = 0,1,2,. . .m, and get

#2 $$\sum_{j=0}^{m} a_j e^j \int_0^j e^{-x}f(x)dx = -\sum_{j=0}^{m} a_jF(j)$$

$$= -\sum_{j=0}^{m} \sum_{i=0}^{mp+p-1} a_j f^i(j).$$

Now the application of the lemma with n replaced by p-1 shows that $f^i(j)$ is an integer for all values of i and j in the above sum. Even more, it shows that $f^i(j)$ is an integer divisible by p except for the single case where j = 0 and i = p-1. A direct calculation from the definition of f(x) establishes that

$$f^{p-1}(0) = (-1)^p(-2)^p \ldots (-m)^p.$$

Thus $f^{p-1}(0)$ is not divisible by p if we choose p>m. Furthermore, if we choose $p>|a_0|$, we see that the right member of (2) consists of a sum of multiples of p with one exception, namely $-a_0 f^{p-1}(0)$. Thus the sum on the right of (2) is a non-zero integer but the left member of (2) satisfies the inequality, by (1),

$$\left|\sum_{j=0}^{m} a_j e^j \int_0^j e^{-x} f(x)dx\right| \leq \Sigma \left|a_j e^j \int_0^j e^{-x} f(x)dx\right|$$

$$\leq \Sigma |a_j| e^j j1m^{mp+p-1}/(p-1)!$$

$$\leq [\Sigma |a_j|]e^m mm^{mp+p-1}/(p-1)!$$

$$\leq [\Sigma |a_j|]e^m (m^{m+2})^{p-1}/(p-1)!$$

$$\leq 1,$$

provided p is chosen sufficiently large. Thus we have a contradiction, and the theorem is proved. [7, pp. 25-27]

Theorem Analysis: The statement #1 in the preceding proof is justified since the max,

$$m^{p-1} > x^{p-1}$$

$$m^p > (x-1)^p$$

$$\vdots$$

$$m^p > (x-m)^p,$$

which implies

$$[m^{mp}][m^{p-1}] = m^{mp+p-1} > | f(x) |.$$

Now,

$$d[e^{-x}F(x)]/dx = e^{-x}[F'(x) – F(x)] = -e^{-x}f(x),$$

is obvious when we recognize

$$F'(x) = f'(x) + f^2(x) + . . . + f^{mp+p-1}(x) + f^{mp+p}(x),$$

where $f^{mp+p}(x) = 0$ since f(x) is a function of x to the mp+p-1 power.

Next, #2 is obtained by the hypothesis that e satisfies the polynomial. That is,

$$\sum_{j=0}^{m} a_j e^j {}_0\int^j e^{-x}f(x)dx = \sum_{j=0}^{m} e^j a_j F(j) - \sum_{j=0}^{m} a_j F(j)$$

$$= [a_m e^m + a_{m-1}e^{m-1} + . . . + a_1 e + a_0][f(0)+f'(0)+. . .+f^{mp+p-1}(0)] - \sum_{j=0}^{m} a_j F(j)$$

$$= [(0)][f(0)+f'(0)+. . .+f^{mp+p-1}(0)] - \sum_{j=0}^{m} a_j F(j)$$

$$= - \sum_{j=0}^{m} a_j F(j)$$

#3 $$= - \sum_{j=0}^{m} \sum_{i=0}^{mp+p-1} a_j f^i(j),$$

since

$$\sum_{j=0}^{mp+p-1} a_j f^i(j) = a_j[f^0(j)+f'(j)+. . .+f^{mp+p-1}(j)]$$

$$= a_j F(j).$$

Now by applying the lemma and selecting p>m and p>$|a_0|$, #3 is the sum of multiples of p except for the term $–a_0f^{p-1}(0)$. This implies #3 is a non-zero integer. The triangle inequality leads to the statement

$$\left|\sum_{j=0}^{m} a_j e^j {}_0\int^j e^{-x}f(x)dx\right| \leq \Sigma \left|a_j e^j {}_0\int^j e^{-x}f(x)dx\right|.$$

This inequality in conjunction with #1 yields,

$$\left|\sum_{j=0}^{m} a_j e^j {}_0\int^j e^{-x} f(x)dx\right| \leq \Sigma \left|a_j e^j {}_0\int^j e^{-x}dx\right| m^{mp+p-1}/(p-1)!.$$

If we observe

$${}_0\int^j e^{-x}dx = e^{-x}\Big]_0^j = (1-1/e^j),$$

and

$$\sum_{j=0}^{m}(1-1/e^j) = (1-1/e^0)+(1-1/e^1)+(1-1/e^2)+...+(1-1/e^m)$$

$$= [0+1+1+1...+1] - [1/e+1/e^2+...+1/e^m]$$

$$= (m-k) < m;$$

then for every j, $(1-1/e^j) \leq j$, where j = 0, 1, 2, . . . m. This implies

$$\left|\sum_{j=0}^{m} a_j e^j {}_0\int^j e^{-x} f(x)dx\right| \leq \Sigma \left|a_j e^j {}_0\int^j e^{-x}dx\right| m^{mp+p-1}/(p-1)!.$$

In addition,

$$\sum_{j=0}^{m} e^j j = 0+e^1+2e^2)+3e^3...+me^m);$$

so for every j, $e^j j \leq me^{m.}$ Consequently,

$$\left|\sum_{j=0}^{m} a_j e^j {}_0\int^j e^{-x} f(x)dx\right| \leq [\Sigma|a_j|]me^m m^{mp+p-1}/(p-1)!$$

$$\leq [\Sigma|a_j|]e^m(m^{m+2})^{p-1}/(p-1)!$$

Lastly, by the ratio test, if p is chosen sufficiently large we have m/p < 1 since p is selected greater than m. Now we have arrived at the statement that the left member of the original inequality is a non-zero integer less than one, an obvious contradiction.

The following theorems are stated without proof and will be used to establish the transcendence of specific number and functions.

THEOREM 2: (Closure property of algebraic numbers)

If F is a subfield of a field E, then relative algebraic closure of A of F is a subfield of E which is algebraic over F and contains all the subfields of E which are algebraic over F. [1, pp. 52-53]

THEOREM 3: (The Generalized Lindermann Theorem) Given and distinct algebraic numbers $a_{1,}\ a_{2,}\ a_{3,}\ldots a_{n,}$ the equation

$$\sum_{j=0}^{m} b_j e^{a_j} = 0$$

are impossible in algebraic numbers $b_{1,}\ b_{2,}\ b_{3,}\ldots b_{n,}$ not all zero. [7, p. 117]

THEOREM 4: (Form A); (The Gelfond-Schneider Theorem)

Let a and b be algebraic numbers different from 0 and 1. If the number

$$n = \log a / \log b$$

is not rational, then it is also transcendental. [9, p. 45]

THEOREM 4: (Form B); If c and d are algebraic numbers different from 0 and 1, and d is not a real rational number, then any value c^d is transcendental. {5, p. 11]

It is necessary at this point to demonstrate the above forms of Theorem 4 are equivalent.

Case I: Form B implies form A.

Proof: Assume the hypothesis of form A and the contrary of the conclusion. That is, n is algebraic but not rational, then

$$n\log b = \log a$$

$$b^n = a.$$

However, this implies a is transcendental by form B. Thus we have a contridiction. [7, p. 135]

Case II: Form A implies form B.

Proof: Assume the hypothesis of form B and the contrary of the conclusion. That is, c^d is algebraic, then

$$n = c^d$$

$$\log n = d\log c$$

$$\log n / \log c = d$$

which implies d is rational of transcendental. Thus we have a contradiction since d is algebraic but not a real rational number.

Before investigating several specific numbers, it should be noted that Theorem 3 affords a very short proof for the transcendence of e. That is, let j = 1 and a = 1, then

$$\sum_{j=0}^{m} b_j e^{a_j} = b_1 e = 0,$$

is impossible in the set of algebraic numbers not all zero. Therefore, by the definition of algegraic numbers, e is transcendental. In addition, this theorem implies e^a is transcendental for all a algebraic.

The closure property of algebraic numbers in conjunction with Theorem 3, makes the proof of the transcendence of π relatively simple.

Proposition 1: π is transcendental.

Proof: Assume π is algebraic. Clearly, i is algebraic since it is a solution of the equation $x^2 + 1 = 0$. Closure implies $i\pi$ is algebraic and Theorem 3 implies $e^{i\pi}$ Is transcendental. However, $e^{i\pi} = -1$, an obvious contradiction. Therefore, π is transcendental. [7, p, 118]

Now, the nature of the trig functions, specifically the tangent and hyperbolic cosine will be considered.

Proposition 2: Tan a, where a is a non-zero algebraic number, is transcendental.

Proof: Assume tan a = c, where c is algebraic. Now by definition,

$$\tan a = [e^{ia} - e^{-ia}]/[ie^{ia} + ie^{-ia}] = c,$$

which implies

$$e^{ia} - e^{-ia} = cie^{ia} + cie^{-ia}.$$

However, this yields

$$(1\text{-}ci)e^{ia} - (1+ci)e^{ia} = 0,$$

where the coefficients and the exponents are algebraic. Thus we have arrived at a contradiction of Theorem 3. Consequently, tan a, where a in a non-zero algebraic number, is transcendental.

Proposition 3: Cosh b, where b is a non-zero algebraic number is transcendental.

Proof: The proof is analogous to that of Proposition 2. Assume cosh b = a, where a is algebraic. By definition of cosh, this leads to the equation

$$e^{b} + e^{-b} - 2ae^{0} = 0,$$

which contradicts Theorem 3.

The remaining trig functions may be proved in a similar manner.

Once again, Lindemann's Theorem offers a simple method to demonstrate the transcendence of a function.

Proposition 4: Log a, where a is a positive algebraic number different from 1, is transcendental.

Proof: Assume Log a = b, where b is an algebraic number.

This implies

$$e^b = a$$

$$a^b - ae^0 = 0,$$

a contradiction of Theorem 3. Therefore, Log a is transcendental.

Numbers such as e^π and $2^{\sqrt{2}}$ remained out of reach until Gelfond's Theorem was presented in 1935. Clearly $2^{\sqrt{2}}$ is transcendental by the irrationality of the $\sqrt{2}$ and a direct application of Gelfond's Theorem. Pollard suggests a similar argument to establish the transcendence of e^π if one observes e^π may be written $i^{-2i.}$ [9, p. 46] Pollard lets the proof to the reader.

Proposition 5: e^π is transcendental.

Proof: From the properties of complex numbers we have

$$e^\pi = [\,(e^\pi)^i]^{1/i}$$

$$= (-1)^{-i}$$

$$= (-1^{1/2})^{-2i}$$

$$= i^{-2i}.$$

Now, i is algebraic and –2i is not a real rational number. Therefore, by application of Theorem 4 (Form B), $i^{-2i} = e^\pi$ is transcendental.

It is now obvious that one cannot classify two of the four types of numbers discussed previously. Namely, a^b and t_1^{t2}. The first because of Gelfond's Theorem and the second because of the following examples.

1. e^π is transcendental
2. $e^{i\pi}$ is algebraic

In order to deny any generalizations concerning numbers of the form t^a, we need only find an example where t^b is algebraic. Lindemann's Theorem guarantees the existence of numbers of the form t^c that are transcendental. Namely, e^a.

Before consideration of the next two propositions, it is essential that we recall a definition and a theorem from complex analysis.

DEFINITION: Let $z \neq 0$ be any complex number. If w is a complex number such that $e^w = z$, then w is called a logarithm of z and will be denoted by w = log z. Furthermore,

$$w = \log z = \mathrm{Log}|z| + i\mathrm{Arg}\, z + 2n\pi,$$

where

$$-\pi < \mathrm{Arg}\, z \leq \pi, \text{ n and integer. } [8, p. 108]$$

THEOREM: If $z \neq 0$, w, and v are any complex numbers,

then

$$\mathrm{Log}(z^w) = w\mathrm{Log}\, z + 2\pi ik,$$

$$(z^w)^v = z^{wv}e^{2\pi ivk},$$

where k is the integer given by the bracket function

$$k = [1/2 - (I(w)\mathrm{Log}|z| + R(w)\mathrm{Arg}\, z)/2\pi]. \quad [8, p. 113]$$

Proposition 6: $[e^{\pi}]^i$ is algebraic.

Proof: The complex properties mentioned above imply

$$(e^{\pi})^i = (e^{\pi i})(e^{2\pi i2k)}$$

$$= (e^{\pi i})(e^{-2\pi 0})$$

$$= (e^{\pi i})(1) = 1,$$

an algebraic number.

The last type of number, those of the form a^t, may or may not be transcendental. The following examples will deny any possible generalization concerning numbers of this form. First, a theorem application in Lang's Introduction to Transcendental Numbers, demonstrates that among the numbers 2^π, 3^π, 5^π, . . ., at most two are algebraic. [5, p. 11] The following proof is an attempt to demonstrate the transcendence of 2^π.

Proposition 7: 2^π is transcendental.

Lemma: If b is a non-zero algebraic number and t is transcendental, then the product bt is transcendental.

Proof: Assume bt is algebraic. This implies there exists a_i's, not all zero, such that

$$a_m(bt)^m + a_{m-1}(bt)^{m-1} + . . . + (ab)t + a_0 = 0.$$

This implies

$$(a_m b^m)t^m + (a_{m-1}b^{m-1})t^{m-1} + . . . + (ab)t + a_0 = 0,$$

where not all $a_i b^i$'s equal zero. This contradicts the hypothesis that t is transcendental. Therefore, bt is transcendental.

Corollary: If t is transcendental, then $t^{-1} = 1/t$, is transcendental.

Proof: Assume there exists a t_1 in the set of transcendentals such that t_1^{-1} is algebraic. This implies the product is transcendental. But,

$$(t_1^{-1})(t_1) = 1,$$

which is an obvious contradiction since 1 is algebraic.

Proof of Proposition 7: Assume 2^π is algebraic. This implies $2^{2\pi}$ is algebraic by the closure property of algebraic numbers. Now,

$$e^{\ln 2^\wedge\pi} = e^{2\pi}$$

which implies the left member is algebraic. Consequently, by Gelfond's Theorem,

$$[e^{\ln 2^\wedge\pi}]^i = [e^{2\pi \ln 2}]^I$$

is transcendental. Now, with the use of the aforementioned complex properties we have

$$[e^{2\pi \ln 2}]^i = [e^{2\pi i \ln 2}][e^{2\pi i^2 k}],$$

where k = 0. Therefore,

$$[e^{2\pi \ln 2}]^i = [e^{2\pi i \ln 2}][e^0]$$

$$= [e^{2\pi i \ln 2}].$$

The corollary to the lemma implies

$$1/[e^{2\pi i \ln 2}] = [e^{2\pi i \ln 2}]^{-1}$$

is transcendental. Therefore,

$$[e^{2\pi i \ln 2}]^{-1} = [e^{-2\pi i \ln 2}]^{-1}[e^{-2\pi i k}]$$

where k = -2. Consequently,

$$[e^{2\pi i \ln 2}]^{-1} = [e^{-2\pi i \ln 2}][e^{4\pi i}]$$

$$= [e^{-2\pi i \ln 2}][1]$$

$$= e^{-2\pi i \ln 2}$$

is transcendental. However,

$$[e^{2\pi i}]^{\ln 2} = [e^{2\pi i \ln 2}][e^{2\pi i k \ln 2}],$$

where k = -2. Therefore,

$$[e^{2\pi i}]^{\ln 2} = [e^{2\pi i \ln 2}][e^{-4\pi i \ln 2}]$$

which implies

$$e^{-2\pi i \ln 2} = [e^{2\pi i}]^{\ln 2}] = 1^{\ln 2} = 1,$$

an obvious contradiction since 1 is algebraic. Therefore, 2^π is transcendental.

Proposition 8: $2^{1/\log 2}$ is algebraic.

Proof: Let

$$2^{1/\log 2} = x.$$

This implies

$$[1/\log 2][\log 2] = \log x,$$

or

$$1 = \log x$$

which implies x= 10, an algebraic number. Therefore, $2^{1/\log 2}$ is algebraic. This example and that of 2^{π}, eliminates the possibility of classifying numbers of the form a^t.

All attempts that were made to establish the nature of e^e were unsuccessful. It is worth noting that the closure property of algebraic numbers does not hold at infinity. That is, e is an infinite sum of algebraic numbers by definition. However, it was shown that e is transcendental.

In conclusion, the research established that generalizations concerning transcendental classes are tedious at best. Proofs of specific examples are extremely difficult and require considerable preparation for each number we consider.

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